On Moving Shapes Through an Orchard

A J Stanley

April 5, 2020

Abstract

Optimisation of constrained planar movement can result in simple problems with ingenious solutions, such as the Kakeya Needle and Moving Sofa Problems. We examine simply-connected regions with free movement in the plane minus the square lattice $\mathbf{Z}^2$, construct a family of such regions with areas approaching π, and propose that this is a universal upper bound.

Let regions that lie in and can be manoeuvred by a sequence of smooth transformations to any point in $\mathbf{R}^2 - \mathbf{Z}^2$ be called *orchard shapes*.

Any solution to the moving sofa problem is also an orchard shape, since a series of mono-directional, right-angled turns can be constructed to navigate a sofa to any point. Gerver's constant therefore gives a lower bound to the maximal area of an orchard shape of $\mu_G = 2.2195...$ [1].

A canonical orchard shape is the unit square. This can be improved on by attaching co-centric circular segments to opposite edges for an area of $1/2 + \pi/4 = 1.2853....$ This is the first in an infinite family $\{R_n\}$ of convex orchard shapes consisting of rectangles capped with circular segments. They are constructed as follows:

For each $n \in \mathbf{Z}, n > 0$, let k_n be the line passing through points $(0, 0)$ and $(1, n-1)$ in the plane and let l_n be the line through $(0, 1)$ and $(1, n)$. Let S_n be the strip bounded by k_n and l_n. Let C_n be the interior of the circle passing through $(0, 0)$, $(1, 0)$, and $(1, n)$. Then R_n is the intersection $S_n \cap C_n$.

For every n, the region R_n in the constructed orientation can slide indefinitely in a diagonal fashion relative to the inherent grid lines. With a rotation about the centre of C_n, it can also be aligned to slide indefinitely in the vertical direction. These transformations can be used to navigate R_n to any point, hence it is an orchard shape.

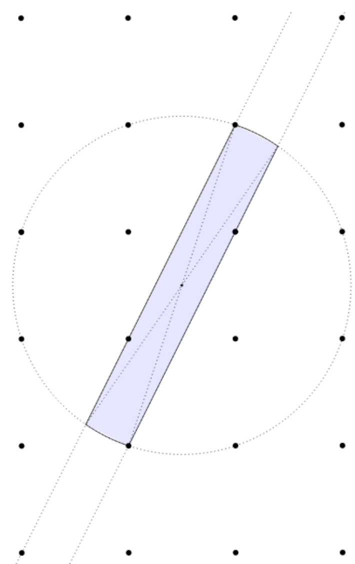

Figure 1: Diagram of R_3 showing the construction lines.

It is not obvious what happens to the areas of these shapes as n increases, but they are finite and we can bound them. Pythagoras tells us the width of S_n is $\sqrt{(n-1)^2+1}^{-1}$ and the diameter of C_n is $\sqrt{n^2+1}$, therefore

$$A(R_n) < \frac{\sqrt{n^2+1}}{\sqrt{(n-1)^2+1}}$$

where $A(R)$ is a function returning the area of region R. Call this upper bound B_n. We have

$$\frac{B_n}{B_{n+1}} = \frac{(\sqrt{n^2+1})^2}{\sqrt{(n-1)^2+1}\sqrt{(n+1)^2+1}} = \frac{\sqrt{n^4+2n+1}}{\sqrt{n^4+4}}$$

Therefore $B_1 < B_2$ and for $n \geqslant 2$ we have $B_n > B_{n+1}$. The exact area of R_2 is given by

$$A(R_2) = \frac{3}{4} + \frac{5}{2} sin^{-1}(\frac{1}{\sqrt{10}}) = 1.5543...$$

which is larger than B_1 and B_3, therefore larger than B_n for all $n \neq 2$. This means $A(R_i) < \frac{\pi}{2}$ for all i. A semi-circle of unit radius, being a solution to the moving sofa problem, is the largest convex orchard shape so far found.

Conjecture 1 *$A(R) \leqslant \pi/2$ for any convex orchard shape R.*

Now consider a second, infinite family of orchard shapes, constructed as follows. First let C_ω be the open disc of radius 1 centred at $(\omega, 0)$ for each $\omega \in \mathbf{R}$. Next choose $\epsilon \in \mathbf{R}$ such that $0 < \epsilon < 1$ and let l_ϵ be the set of points $\{(x,0) : x \leqslant 0 \vee x \geqslant \epsilon\}$. Then

$$D_\epsilon = C_0 \cap C_\epsilon - l_\epsilon$$

is an orchard shape. Since $A(C_0) = A(C_\omega) = \pi$, it is clear that

$$A(D_\epsilon) < \pi \quad \text{and} \quad \lim_{\epsilon \to 0} A(D_\epsilon) = \pi$$

so choosing ϵ sufficiently small will give a region with area greater than μ_G.

Conjecture 2 *$A(R) < \pi$ for any orchard shape R.*

We will consider a final family of shapes $\{T_n\}$ of which not all are orchard shapes. For $n \in \mathbf{Z}, n > 0$, let D_n and D'_n be concentric discs such that D_n passes through $(0,1)$, $(n,0)$, and $(-n,0)$ and D'_n passes through $(n-1,0)$. Their centre will be at $(0,c)$, where $c = (1-n^2)/2$. Let S_1 be the unit strip as defined above. Then T_n is the interior of

$$S_1 \cap (D_n - D'_n)$$

Note that c is an integer whenever n is odd.

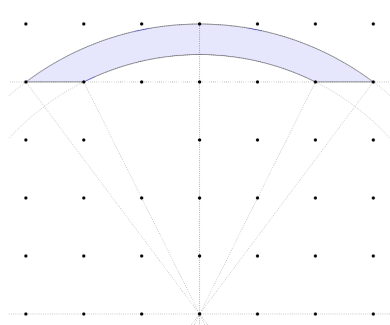

Figure 2: Diagram of T_3 showing the construction lines.

T_1 is a semi-circle of unit radius and has area $\pi/2$. To find the area of T_n for $n > 1$, we will subtract the circular segment $D'_n \cap S_1$ from $D_n \cap S_1$. Let the angles that these segments span at the shared centre of D'_n and D_n

be 2θ and 2Θ respectively. Let R be the radius of D_n and r be the radius of D'_n. Then the area of T_n is given by

$$\begin{aligned} A(T_n) &= \left(\Theta R^2 - n(R-1)\right) - \left(\theta r^2 - (n-1)(R-1)\right) \\ &= \Theta R^2 - \theta r^2 - R + 1 \end{aligned}$$

We can make the following substitutions:

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{n-1}{R-1}\right) \\ \Theta &= \tan^{-1}\left(\frac{n}{R-1}\right) \\ r^2 &= R^2 - 2n + 1 \\ R &= \frac{n^2+1}{2} \end{aligned}$$

to get (for $n > 1$)

$$\begin{aligned} A(T_n) = \tan^{-1}\left(\frac{2n}{n^2-1}\right)\left(\frac{n^2+1}{2}\right)^2 - \\ \tan^{-1}\left(\frac{2n-2}{n^2-1}\right)\left(\left(\frac{n^2+1}{2}\right)^2 - 2n + 1\right) - \frac{n^2-1}{2} \end{aligned} \tag{1}$$

Computing this area for some values, we see $A(T_n)$ increase with n:

$$\begin{aligned} A(T_2) &= 2.3845... \\ A(T_3) &= 2.8145... \\ A(T_4) &= 3.0713... \\ A(T_5) &= 3.2396... \end{aligned}$$

Already the area of T_5 is greater than π. It can be shown that (1) is strictly increasing for $n > 1$ and asymptotically approaches an upper bound of 4. If any of $T_5, T_6, ...$ are orchard shapes then they would be larger than any we have seen so far.

T_n can already be translated indefinitely in the horizontal direction. If $D_n - D'_n$ lies in $\mathbf{R}^2 - \mathbf{Z}^2$ then T_n can be smoothly rotated about $(0, c)$ by $\pi/2$ such that the bounding unit strip is vertical, and if x is an integer then this strip contain no points in $\mathbf{Z}^2$, allowing T_n to be translated indefinitely in the vertical direction. Thus, a sufficient condition for T_n to be an orchard shape for odd n is that the following inequalities have no integer solutions in x and y.

$$R^2 - 2n + 1 < x^2 + y^2 < R^2$$

where $R = \frac{n^2+1}{2}$.

For $n = 1$ we have $R = 1$ and the requirement that no sums of two squares lie between 0 and 1. This is satisfied, therefore T_1 is an orchard shape (but we knew this already). Similarly, $n = 3$ gives $R = 5$ and bounds of 20 and 25. There are no sums of two squares lying between 20 and 25, so T_3 is also an orchard shape. It has an area of 2.8145... which beats Gerver's sofa but not D_ϵ for sufficiently small ϵ. One more and we will have a new record holder, but sadly they dry up after this point.

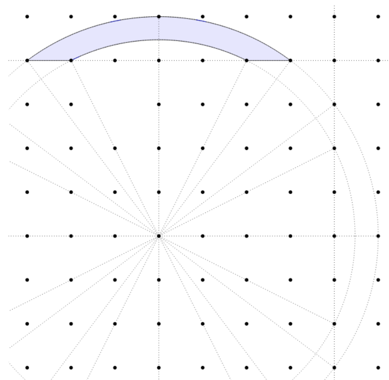

Figure 3: Diagram of T_3 and the path it traces during rotation.

Note that R^2 and $r^2 = R^2 - 2n + 1$ are themselves sums of two squares. The distance function from a given k to the nearest sum of two squares is bounded by $\sqrt[4]{k}$ for $k \geqslant 4$ [2], so there will be sums of two squares between r^2 and R^2 whenever $R^2 - r^2 > 2\sqrt{R} + 1$. In terms of n, this reduces to $n^2 - 4n + 1 > 0$, which is true for all $n \geqslant 4$.

It is unknown whether there exist orchard shapes in $\{T_n\}$ other than the two already mentioned, but none satisfy the given conditions. Some may be orchard shapes despite this, following a more interesting movement pattern as they traverse the world.

References

[1] J. L. Gerver. *On moving a sofa around a corner*. Geometriae Dedicata 42 (1992), 267-283.

[2] R. P. Bambah, Chowla, S. *On numbers which can be expressed as a sum of two squares*. Proc. Nat. Acad. Sci. India 13 (1947), 101-103.